

ON C -SMALL CONJUGACY CLASSES IN A REDUCTIVE GROUP

G. LUSZTIG

INTRODUCTION

0.1. Let G be a connected reductive algebraic group over an algebraically closed field \mathbf{k} of characteristic p . Let \mathbf{W} be the Weyl group of G . Let Ω_w be the double coset in G (with respect to a Borel subgroup B^*) corresponding to an element $w \in \mathbf{W}$ which has minimal length in its conjugacy class C in \mathbf{W} and has no eigenvalue 1 in the reflection representation of \mathbf{W} . Let Z_G be the centre of G . Let γ be a conjugacy class in G such that $\Omega \cap \gamma \neq \emptyset$. Since $\Omega_w \cap \gamma$ is B^*/Z_G -stable for the conjugation action of B^*/Z_G and this action has finite isotropy groups by [L5, 5.2], we see that $\dim(\Omega_w \cap \gamma) \geq \dim(B^*/Z_G)$. As in [L5] we say that γ is C -small if the previous inequality is an equality. (This condition depends only on C , not on w .)

In the remainder of this subsection we assume that

(i) p is 0 or a good prime for G

and that G is almost simple. In [L5] we have shown that for any C as above there is a unique unipotent class γ_C in G which is C -small. In this paper we investigate the existence of C -small semisimple classes in G . We show that such a class γ' exists in almost all cases. (There is exactly one exception to this property: it arises in type E_8 for a unique C .) Let ρ_{γ_C} be the Springer representation of \mathbf{W} associated to γ_C and to the local system \mathbf{Q}_l on γ_C . We show that ρ_{γ_C} is surprisingly connected to γ' above (again with the unique exception above) as follows: ρ_{γ_C} is obtained by "j-induction" (see 0.3) from the sign representation of a reflection subgroup of \mathbf{W} , namely the Weyl group of the connected centralizer of an element of γ' . We will also show that the representation ρ_{γ_C} depends only on the Weyl group \mathbf{W} , not on the underlying root system.

0.2. Here is some notation that we use in this paper. Let \mathcal{B} the variety of Borel subgroups of G . Let $\underline{l} : \mathbf{W} \rightarrow \mathbf{N}$ be the standard length function. Let $S = \{s \in \mathbf{W}; \underline{l}(s) = 1\}$. For each $w \in \mathbf{W}$ let \mathcal{O}_w be the corresponding G -orbit in $\mathcal{B} \times \mathcal{B}$. Let $\underline{\mathbf{W}}_{el}$ be the set of elliptic conjugacy classes in \mathbf{W} (see [L5, 0.2].) For $C \in \underline{\mathbf{W}}_{el}$ let $d_C = \min_{w \in C} \underline{l}(w)$ and let $C_{min} = \{w \in C; \underline{l}(w) = d_C\}$. For any conjugacy

Supported in part by the National Science Foundation

class γ in G and any $w \in \mathbf{W}$ we set $\mathfrak{B}_w = \{(g, B) \in G \times \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}$, $\mathfrak{B}_w^\gamma = \{(g, B) \in \mathfrak{B}_w; g \in \gamma\}$.

The cardinal of a finite set X is denoted by $|X|$ or by $\sharp(X)$. For any $g \in G$ let $Z(g)$ be the centralizer of g in G . Let ν_G be the number of positive roots of G . For an integer σ we define $\kappa_\sigma \in \{0, 1\}$ by $\sigma = \kappa_\sigma \pmod{2}$.

Let $C \in \underline{\mathbf{W}}_{el}$. In [L5, 5.5, 5.7(iii)] it is shown that if $\mathfrak{B}_w^\gamma \neq \emptyset$ for some/any $w \in C_{min}$ then $\dim \mathfrak{B}_w^\gamma \geq \dim(G/Z_G)$ and $\dim \gamma \geq \dim(G/Z_G) - d_C$. (Here the equivalence of "some/any" follows from [L5, 5.2(a)].) Following [L5, 5.5] we say that γ is *C-small* if for some/any $w \in C_{min}$ we have $\mathfrak{B}_w^\gamma \neq \emptyset$ and the equivalent conditions $\dim \mathfrak{B}_w^\gamma = \dim(G/Z_G)$, $\dim \gamma = \dim(G/Z_G) - d_C$ are satisfied (for the equivalence see [L5, 7.7(iv)]).

0.3. Let W be a Weyl group. Let sgn be the sign representation of W and let R_W be the reflection representation of W . Let $\text{Irr}(W)$ be the set of (isomorphism classes of) irreducible representations of W . For $E \in \text{Irr}(W)$ let b_E be the smallest integer ≥ 0 such that the multiplicity of E in the b_E -th symmetric power of R_W is ≥ 1 . We write $E \in \text{Irr}(W)^\dagger$ if this multiplicity is 1. Let W' be a subgroup of W generated by reflections. Let $E' \in \text{Irr}(W')^\dagger$. There is a unique $E \in \text{Irr}(W)$ such that E appears in $\text{ind}_{W'}^W(E')$ and $b_E = b_{E'}$. (See [GP, 5.2.6].) We have $E \in \text{Irr}(W)^\dagger$. We set $E = j_{W'}^W(E')$. The process $j_{W'}^W(\cdot)$ is called *j-induction*.

0.4. For any unipotent class γ in G let $\rho_\gamma \in \text{Irr}(\mathbf{W})$ be the Springer representation of \mathbf{W} associated to γ and the local system $\bar{\mathbf{Q}}_l$ on γ . (We use the conventions of [L2].) For any $C \in \underline{\mathbf{W}}_{el}$ let γ_C be the unique *C-small* unipotent class of G , see [L5]; thus ρ_{γ_C} is well defined.

0.5. Let B^* be a Borel subgroup of G and let \mathcal{T} be a maximal torus of B^* . Let $N_G(\mathcal{T})$ be the normalizer of \mathcal{T} in G and let $\mathcal{W} = N_G(\mathcal{T})/\mathcal{T}$. For any $z \in \mathcal{W}$ let \dot{z} be a representative of z in $N_G(\mathcal{T})$. We identify $\mathbf{W} = \mathcal{W}$ as follows: to $z \in \mathcal{W}$ corresponds the element $w \in \mathbf{W}$ such that $(B^*, \dot{z}B^*\dot{z}^{-1}) \in \mathcal{O}_w$. For any $s \in S$ let $\alpha_s : \mathcal{T} \rightarrow \mathbf{k}^*$ be the simple root defined by s . In the remainder of this subsection we assume that G is almost simple, simply connected and that 0.1(i) holds. Let $\alpha_0 : \mathcal{T} \rightarrow \mathbf{k}^*$ be the unique root such that for any $s \in S$, $\alpha_0 \alpha_s^{-1} : \mathcal{T} \rightarrow \mathbf{k}^*$ is not a root. Let $\Delta = \{\alpha_s; s \in S\} \sqcup \{\alpha_0\}$. For any $K \subsetneq \Delta$ let \mathcal{W}_K be the subgroup of \mathcal{W} generated by the reflections with respect to roots in Δ . Let G_K be the subgroup of G generated by \mathcal{T} and by the root subgroups attached to roots such that the corresponding reflection in \mathcal{W} is in \mathcal{W}_K . Note that G_K is a connected reductive subgroup of G (a "Borel-de Siebenthal subgroup"). Now $B^* \cap G_K$ is a Borel subgroup of G_K and \mathcal{T} is a maximal torus of $B^* \cap G_K$. Note that $\mathcal{W}_K = N_{G_K}(\mathcal{T})/\mathcal{T} = \{w \in \mathcal{W}; w \in G_K\}$ may be identified (using $B^* \cap G_K$, \mathcal{T}) with the Weyl group \mathbf{W}_K of G_K in the same way as \mathcal{W} is identified with \mathbf{W} (using B^* , \mathcal{T}). In particular \mathbf{W}_K appears as a subgroup of \mathbf{W} . Let \mathcal{S}_K be the set of semisimple conjugacy classes γ in G such that for some $\zeta \in \gamma \cap \mathcal{T}$ we have $G_K = Z(\zeta)$. Note that $\mathcal{S}_K \neq \emptyset$ and any semisimple class in G belongs to \mathcal{S}_K for some $K \subsetneq \Delta$. The following is our main result.

Theorem 0.6. *Assume that G is almost simple, simply connected and that 0.1(i) holds. Let $C \in \underline{\mathbf{W}}_{el}$. With the single exception when G is of type E_8 and for any $w \in C$, the characteristic polynomial of $w : R_{\mathbf{W}} \rightarrow R_{\mathbf{W}}$ is $(X+1)(X^2+1)^2(X^3+1)$, there exists $K \subsetneq \Delta$ such that*

- (i) *for any $\gamma \in \mathcal{S}_K$, γ is a C -small semisimple class;*
- (ii) $\rho_{\gamma_C} = j_{\mathbf{W}_K}^{\mathbf{W}}(\text{sgn})$.

In the case where G is of type E_8 and C is the class specified in the theorem, there is no $K \subsetneq \Delta$ for which (i) holds and there is no $K \subsetneq \Delta$ for which (ii) holds. On the other hand in this case we have $\rho_{\gamma_C} = j_{\mathbf{W}_K}^{\mathbf{W}}(\text{sgn} \otimes r)$ where G_K is of type $D_5 + A_3$ and r is the irreducible representation of \mathbf{W}_K on which the D_5 -factor acts as the reflection representation and the A_3 -factor acts trivially. Also, if $\gamma \in \mathcal{S}_K$, $\zeta \in \gamma \cap \mathcal{T}$, $Z(\zeta) = K$ and u is a unipotent element of G_K which is in a minimal unipotent class $\neq 1$ of the D_5 -factor, then the G -conjugacy class γ' of ζu is C -small; although γ' is not semisimple, it is as close as possible to being semisimple.

In the case where G is of type A the theorem is immediate: C must be the conjugacy class of a Coxeter element and we can take $K = \emptyset$. The proof of the theorem in the case where G is of classical type other than A is given in §1, §3. When G is of exceptional type the proof of the theorem is given in 2.4, 3.5 (using a reduction to a computer calculation, see 2.2.)

0.7. Assume that G is almost simple, simply connected and that 0.1(i) holds. Let γ be any C -small conjugacy class in G . Let ζ (resp. u) be a semisimple (resp. unipotent) element of G such that $\zeta u = u\zeta \in \gamma$. Let $K \subsetneq \Delta$ be such that the conjugacy class of ζ belongs to \mathcal{S}_K . We assume as we may that $\zeta \in \gamma \cap \mathcal{T}$ and $G_K = Z(\zeta)$. Let ρ_u be the Springer representation of \mathbf{W}_K associated to the conjugacy class of u in G_K . We conjecture that

$$\rho_{\gamma_C} = j_{\mathbf{W}_K}^{\mathbf{W}}(\rho_u).$$

This is supported by Theorem 0.6.

0.8. I thank Gongqin Li for her help with programming in GAP3.

1. CLASSICAL GROUPS

1.1. Let t_1, t_2, \dots, t_m be commuting indeterminates. Let M be the $m \times m$ matrix whose j -th row is $t_1^j - t_1^{-j}, t_2^j - t_2^{-j}, \dots, t_m^j - t_m^{-j}$, $j \in [1, m]$. Let M' be the $(m+2) \times (m+2)$ matrix whose j -th row is $1, (-1)^j, t_1^j + t_1^{-j}, t_2^j + t_2^{-j}, \dots, t_m^j + t_m^{-j}$, $j \in [0, m+1]$. The proof of (a), (b) below is left to the reader.

(a) $\det(M)$ is equal to $\pm \prod_i \prod_{i < j} (t_i - t_j) \prod_{i \leq j} (t_i - t_j^{-1})$ times a monomial in the t_i ;

(b) $\det(M')$ is equal to $\pm 2 \prod_i (t_i - t_i^{-1}) \prod_{i < j} (t_i - t_j) \prod_{i \leq j} (t_i - t_j^{-1})$ times a monomial in the t_i .

1.2. Let V be a \mathbf{k} -vector space of finite dimension $\mathbf{n} \geq 3$. We set $\kappa = \kappa_{\mathbf{n}}$ so that $\mathbf{n} = 2n + \kappa$ with $n \in \mathbf{N}$. Assume that V has a fixed bilinear form $(,) : V \times V \rightarrow \mathbf{k}$ and a fixed quadratic form $Q : V \rightarrow \mathbf{k}$ such that (i) or (ii) below holds:

(i) $Q = 0$, $(x, x) = 0$ for all $x \in V$, $(,)$ is nondegenerate;

(ii) $Q \neq 0$, $(x, y) = Q(x + y) - Q(x) - Q(y)$ for $x, y \in V$, $p \neq 2$, $(,)$ is nondegenerate.

An element $g \in GL(V)$ is said to be an isometry if $(gx, gy) = (x, y) = 0$ for all $x, y \in V$ (hence $Q(gx) = Q(x)$ for all $x \in V$). Let $Is(V)$ be the group of all isometries of V (a closed subgroup of $GL(V)$). In this section we assume that G is the identity component of $Is(V)$. Let \mathcal{F} be the set of all sequences $V_* = (0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_{\mathbf{n}} = V)$ of subspaces of V such that $\dim V_i = i$ for $i \in [0, \mathbf{n}]$, $Q|_{V_i} = 0$ and $\{x \in V; (x, V_i) = 0\} = V_{\mathbf{n}-i}$ for all $i \in [0, n]$. Now $Is(V)$ acts naturally (transitively) on \mathcal{F} .

1.3. Assume that $Q = 0$ so that $\mathbf{n} = 2n$. Let $p_* = (p_1 \geq p_2 \geq \dots \geq p_{\sigma})$ be a sequence of integers ≥ 1 such that $p_1 + p_2 + \dots + p_{\sigma} = n$. For any $i \geq 1$ we set $\bar{p}_i = \sharp(t \in [1, \sigma]; p_t \geq i)$ so that $\bar{p}_1 \geq \bar{p}_2 \geq \dots$ and $\sum_i \bar{p}_i = n$. Let $k = p_1$. We have $\bar{p}_k \geq 1$, $\bar{p}_{k+1} = 0$. We can find subspaces $\mathcal{V}_i, \mathcal{V}'_i$ ($i \in [1, k]$) of V such that

$$V = \mathcal{V}_1 \oplus \mathcal{V}'_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}'_2 \oplus \dots \oplus \mathcal{V}_k \oplus \mathcal{V}'_k;$$

$$\dim \mathcal{V}_i = \dim \mathcal{V}'_i = \bar{p}_i \text{ for } i \in [1, k];$$

$$(,) \text{ is zero on } \mathcal{V}_i, \mathcal{V}'_i \text{ for } i \in [1, k];$$

$$(\mathcal{V}_i \oplus \mathcal{V}'_i, \mathcal{V}_j \oplus \mathcal{V}'_j) = 0 \text{ for all } i \neq j.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be a sequence elements of \mathbf{k}^* such that

$$\lambda_i - \lambda_j \neq 0 \text{ for } i \neq j,$$

$$\lambda_i - \lambda_j^{-1} \neq 0 \text{ for all } i, j.$$

For any $t \in [1, \sigma]$ the system of linear equations

$$\sum_{i \in [1, p_t]} (\lambda_i^j - \lambda_i^{-j}) c_{t,i} = -\delta_{j, p_t}$$

($j \in [1, p_t]$) with unknowns $c_{t,i}$ ($i \in [1, p_t]$) has a unique solution $(c_{t,i})_{i \in [1, p_t]} \in \mathbf{k}^{p_t}$. (Its determinant is nonzero by 1.1(a) with $m = p_t$. Note that λ_i is defined for $i \in [1, p_t]$ since $p_t \leq p_1 = k$.) For any $i \in [1, k]$ we choose a basis $(v_{t,i})_{t \in [1, \sigma]; p_t \geq i}$ of \mathcal{V}_i and we define vectors $v'_{t',i} \in \mathcal{V}'_i$ ($t' \in [1, \sigma], p_{t'} \geq i$) by $(v_{t,i}, v'_{t',i}) = \delta_{t,t'} c_{t,i}$ for all $t \in [1, \sigma], p_t \geq i$. Then for any $t \in [1, \sigma]$ and any $j \in [1, p_t]$ we have

$$(a) \sum_{i \in [1, p_t]} (\lambda_i^j - \lambda_i^{-j}) (v_{t,i}, v'_{t,i}) = -\delta_{j, p_t}.$$

Hence for $j \in [-p_t, p_t - 1]$ we have

$$(b) \sum_{i \in [1, p_t]} (\lambda_i^j - \lambda_i^{-j}) (v_{t,i}, v'_{t,i}) = \delta_{j, -p_t}.$$

(For $j \in [1, p_t - 1]$, (b) follows from (a); for $j \in [-p_t, -1]$, (b) follows from (a) by replacing j by $-j$; for $j = 0$, (b) is obvious.) For any $t \in [1, \sigma]$ we set $v_t = \sum_{i \in [1, k]; i \leq p_t} (v_{t,i} + v'_{t,i}) \in V$. Define a linear map $g : V \rightarrow V$ by $gx = \lambda_i x$ for $x \in \mathcal{V}_i$, $gx = \lambda_i^{-1} x$ for $x \in \mathcal{V}'_i$ ($i \in [1, k]$). Then $g \in G$.

1.4. Assume that $Q \neq 0$ so that $p \neq 2$. Let $p_* = (p_1 \geq p_2 \geq \cdots \geq p_\sigma)$ be a sequence of integers ≥ 1 such that $p_1 + p_2 + \cdots + p_\sigma = n$. If $\kappa = 0$ we assume also that $\kappa_\sigma = 0$. For any $i \geq 1$ we set $\bar{p}_i = \sharp(t \in [1, \sigma]; p_t \geq i)$ so that $\bar{p}_1 \geq \bar{p}_2 \geq \cdots$ and $\sum_i \bar{p}_i = n$. Let $k = p_1$. We have $\bar{p}_1 = \sigma$, $\bar{p}_k \geq 1$, $\bar{p}_{k+1} = 0$.

We can choose subspaces $\mathcal{Z}', \mathcal{Z}''$ of V and subspaces $\mathcal{V}_i, \mathcal{V}'_i$ of V (for $i \in [2, k]$) such that:

$$\begin{aligned} V &= \mathcal{Z}' \oplus \mathcal{Z}'' \oplus \bigoplus_{i \in [2, k]} (\mathcal{V}_i \oplus \mathcal{V}'_i); \\ (,) &\text{ is nondegenerate on } \mathcal{Z}' \text{ and on } \mathcal{Z}'', (,) \text{ is zero on } \mathcal{V}_i, \mathcal{V}'_i \text{ (for } i \in [2, k]); \\ \dim \mathcal{Z}' &= \bar{p}_1 + \kappa - \kappa_\sigma, \dim \mathcal{Z}'' = \bar{p}_1 + \kappa_\sigma; \\ \dim \mathcal{V}_i &= \dim \mathcal{V}'_i = \bar{p}_i \text{ for } i \in [2, k]; \\ (\mathcal{V}_i \oplus \mathcal{V}'_i, \mathcal{V}_j \oplus \mathcal{V}'_j) &= 0 \text{ for } i \neq j \text{ in } [2, k]; \\ (\mathcal{Z}', \mathcal{Z}'') &= 0; \\ (\mathcal{Z}' + \mathcal{Z}'', \mathcal{V}_i + \mathcal{V}'_i) &= 0 \text{ for } i \in [2, k]. \end{aligned}$$

Let $\lambda_i (i \in [2, k])$ be elements of \mathbf{k}^* such that

$$\begin{aligned} \lambda_i - \lambda_j &\neq 0 \text{ for } i \neq j \text{ in } [2, k], \\ \lambda_i - \lambda_j^{-1} &\neq 0 \text{ for all } i, j \text{ in } [2, k]. \end{aligned}$$

For any $t \in [1, \sigma]$, the system of linear equations

$$c_{t,1} + (-1)^j c_{t,-1} + \sum_{i \in [2, p_t]} (\lambda_i^j + \lambda_i^{-j}) c_{t, \lambda_i} = \delta_{j, p_t}$$

($j \in [0, p_t]$) with $(p_t + 1)$ unknowns ($c_{t, \lambda_i} \in \mathbf{k}$ ($i \in [2, p_t]$) and $c_{t,1} \in \mathbf{k}, c_{t,-1} \in \mathbf{k}$) has a unique solution. (Its determinant is nonzero by 1.1(b) with $m = p_t - 1$.) For any $i \in [2, k]$ we choose a basis $(v_{t,i})_{t \in [1, \sigma]; p_t \geq i}$ of \mathcal{V}_i and we define vectors $v'_{t',i} \in \mathcal{V}'_i$ ($t' \in [1, \sigma], p_{t'} \geq i$) by $(v_{t,i}, v'_{t',i}) = \delta_{t,t'} c_{t, \lambda_i}$ for all $t \in [1, \sigma], p_t \geq i$. (Note that if $t \in [1, \sigma]$ is such that $p_t \geq i$ then $i \in [2, p_t]$ hence c_{t, λ_i} is defined.)

We can find vectors $v'_t \in \mathcal{Z}'$ ($t \in [1, \sigma]$) such that $(v'_t, v'_{t'}) = c_{t,1} \delta_{t,t'}$ for all t, t' in $[1, \sigma]$. We can find vectors $v''_t \in \mathcal{Z}''$ ($t \in [1, \sigma]$) such that $(v''_t, v''_{t'}) = c_{t,-1} \delta_{t,t'}$ for all t, t' in $[1, \sigma]$. Then for any $t \in [1, \sigma]$ and any $j \in [0, p_t]$ we have

$$(v'_t, v'_t) + (-1)^j (v''_t + v''_t) + \sum_{i \in [2, p_t]} (\lambda_i^j + \lambda_i^{-j}) (v_{t,i}, v'_{t,i}) = \delta_{j, p_t}.$$

It follows that

$$(v'_t, v''_t) + (-1)^j (v''_t, v''_t) + \sum_{i \in [2, p_t]} (\lambda_i^j + \lambda_i^{-j}) (v_{t,i}, v'_{t,i}) = \delta_{-j, p_t}$$

for $j \in [-p_t, p_t - 1]$. For any $t \in [1, \sigma]$ we set $v_t = v'_t + v''_t + \sum_{i \in [2, k]; i \leq p_t} (v_{t,i} + v'_{t,i}) \in V$. Define a linear map $g : V \rightarrow V$ by $gx = x$ for $x \in \mathcal{Z}'$, $gx = -x$ for $x \in \mathcal{Z}''$, $gx = \lambda_i x$ for $x \in \mathcal{V}_i$ ($i \in [2, k]$), $gx = \lambda_i^{-1} x$ for $x \in \mathcal{V}'_i$ ($i \in [2, k]$). Then $g \in G$.

1.5. Assume that we are in the setup of 1.3 or 1.4. For t, t' in $[1, \sigma]$ and $j \in [-p_t, p_t - 1]$ we have

$$(a) \quad (g^j v_t, v_{t'}) = \delta_{t,t'} \delta_{-j, p_t}.$$

Indeed, in the setup of 1.3, the left hand side of (a) is equal to

$$\begin{aligned} & \sum_{i \in [1, k]; i \leq p_t, i \leq p_{t'}} (\lambda_i^j v_{t,i} + \lambda_i^{-j} v'_{t,i}, v_{t',i} + v'_{t',i}) \\ &= \delta_{t,t'} \sum_{i \in [1, p_t]} (\lambda_i^j - \lambda_i^{-j}) (v_{t,i}, v'_{t,i}) = \delta_{t,t'} \delta_{j, -p_t}; \end{aligned}$$

in the setup of 1.4, the left hand side of (a) is equal to

$$\begin{aligned} & (v_{t,1}, v_{t',1}) + (-1)^j (v'_{t,1}, v'_{t',1}) + \sum_{i \in [2, k]; i \leq p_t, i \leq p_{t'}} (\lambda_i^j v_{t,i} + \lambda_i^{-j} v'_{t,i}, v_{t',i} + v'_{t',i}) \\ &= \delta_{t,t'} ((v_{t,1}, v_{t,1}) + (-1)^j (v'_{t,1}, v'_{t,1})) \\ &+ \sum_{i \in [2, k]; i \leq p_t, i \leq p_{t'}} (\lambda_i^j + \lambda_i^{-j}) (v_{t,i}, v'_{t,i}) = \delta_{t,t'} \delta_{j, -p_t}. \end{aligned}$$

As in [L5, 3.3(vi)], from (a) we deduce that the vectors $(g^j v_t)_{t \in [1, \sigma], j \in [-p_t, p_t - 1]}$ span a $(\mathbf{n} - \kappa)$ -dimensional subspace of V on which $(,)$ is nondegenerate. (If $\kappa = 0$ this subspace is V .) For any $h \in [1, n]$ we can write $h = p_1 + p_2 + \cdots + p_{r-1} + i$ where $r \in [1, \sigma]$ and $i \in [1, p_r]$ are uniquely determined; we define V_h to be the subspace of V spanned by the vectors $g^j v_t (t \in [1, r-1], j \in [0, p_t - 1])$ and $g^j v_r (j \in [0, i - 1])$. Let V'_h be the subspace of V spanned by the vectors $g^j v_t (t \in [1, r-1], j \in [1, p_t])$ and $g^j v_r (j \in [1, i])$. We have $(V_h, V_h) = 0$ (see (a)), $(V'_h, V'_h) = 0$, $gV_h = V'_h$. There are unique sequences V_*, V'_* in \mathcal{F} such that V_h, V'_h are as above for any $h \in [1, n]$. We have $V'_* = gV_*$. For any $r \in [1, \sigma]$ and $i \in [1, p_r - 1]$ we have

$$\dim(V'_{p_1 + p_2 + \cdots + p_{r-1} + i} \cap V_{p_1 + p_2 + \cdots + p_{r-1} + i}) = p_1 + p_2 + \cdots + p_{r-1} + i - r,$$

(the intersection is spanned by the vectors $g^j v_t (t \in [1, r-1], j \in [1, p_t - 1])$ and $g^j v_r (j \in [1, i - 1])$),

$$\dim(V'_{p_1 + p_2 + \cdots + p_{r-1} + i} \cap V_{p_1 + p_2 + \cdots + p_{r-1} + i + 1}) = p_1 + p_2 + \cdots + p_{r-1} + i - r + 1,$$

(the intersection is spanned by the vectors $g^j v_t (t \in [1, r-1], j \in [1, p_t - 1])$ and $g^j v_r (j \in [1, i])$). For any $r \in [1, \sigma]$ we have

$$\dim(V'_{p_1 + p_2 + \cdots + p_r} \cap V_{\mathbf{n} - p_1 - p_2 - \cdots - p_{r-1} - 1}) = p_1 + p_2 + \cdots + p_r - r,$$

(the intersection is spanned by the vectors $g^j v_t (t \in [1, r], j \in [1, p_t - 1])$),

$$\dim(V'_{p_1+p_2+\dots+p_r} \cap V_{\mathbf{n}-p_1-p_2-\dots-p_{r-1}}) = p_1 + p_2 + \dots + p_r - r + 1,$$

(the intersection is spanned by the vectors $g^j v_t (t \in [1, r], j \in [1, p_t - 1])$ and $g^{p_r} v_r$). (We use again (a).) As in [L5, 3.2] we deduce that $a_{V_*, V'_*} = w_{p_*} \in \mathbf{W}$ (notation of [L5, 1.4, 1.6]). Let B, B' be the stabilizers of V_*, V'_* in G . Then B, B' are Borel subgroups of G and $(B, B') \in \mathcal{O}_{w_{p_*}}$, $gBg^{-1} = B'$. Hence if γ denotes the conjugacy class of g in G , we have

$$(b) \quad \mathfrak{B}_{w_{p_*}}^\gamma \neq \emptyset.$$

Note that γ is a semisimple conjugacy class and that w_{p_*} has minimal length in its conjugacy class C in \mathbf{W} (which is elliptic).

Let $\delta(g) = \dim Z(g)$. Let $d = \underline{l}(w_{p_*})$. We show that

$$(c) \quad \delta(g) = d.$$

In the setup of 1.3, $Z(g)$ is isomorphic to $GL(\bar{p}_1) \times GL(\bar{p}_2) \times \dots \times GL(\bar{p}_k)$ hence $\delta(g) = \bar{p}_1^2 + \bar{p}_2^2 + \dots + \bar{p}_k^2$. In the setup of 1.4, the identity component of $Z(g)$ is isomorphic to $SO(\bar{p}_1 + \kappa - \kappa_\sigma) \times SO(\bar{p}_1 + \kappa_\sigma) \times GL(\bar{p}_2) \times GL(\bar{p}_3) \times \dots \times GL(\bar{p}_k)$ hence

$$\begin{aligned} \delta(g) &= (\bar{p}_1 + \kappa - \kappa_s)(\bar{p}_1 + \kappa - \kappa_\sigma - 1)/2 + (\bar{p}_1 + \kappa_s)(\bar{p}_1 + \kappa_s - 1)/2 \\ &\quad + \bar{p}_2^2 + \dots + \bar{p}_k^2 = \bar{p}_1^2 + \bar{p}_2^2 + \dots + \bar{p}_k^2 - \sigma(1 - \kappa). \end{aligned}$$

If $(1 - \kappa)Q = 0$ we have $d = 2(p_2 + 2p_3 + \dots + (\sigma - 1)p_\sigma) + n$; if $(1 - \kappa)Q \neq 0$ we have $d = 2(p_2 + 2p_3 + \dots + (\sigma - 1)p_\sigma) + n - \sigma$. Hence to prove (c) it is enough to show that

$$\bar{p}_1^2 + \bar{p}_2^2 + \dots + \bar{p}_k^2 = 2(p_2 + 2p_3 + \dots + (\sigma - 1)p_\sigma) + n.$$

This follows from the equality $X = 2Y$ in [L5, 4.4]; note that f_{2h} from *loc.cit.* is the same as \bar{p}' and $\sum_h f_{2h} = n$. From (b) and (c) we see that γ is a (semisimple) C -small conjugacy class. This proves 0.6(i) for our G .

2. EXCEPTIONAL GROUPS

2.1. In this subsection we assume that \mathbf{k} is an algebraic closure of a finite field \mathbf{F}_q with q elements; we also assume that 0.1(i) holds. We choose an \mathbf{F}_q -split rational structure on G with Frobenius map $F : G \rightarrow G$ such that B^* and \mathcal{T} are F -stable. Note that $F(t) = t^q$ for all $t \in \mathcal{T}$. Define a class function $\Pi_G : \mathbf{W} \rightarrow \mathbf{Z}$ by $\Pi_G(w) = \sum_i \text{tr}(w, H^{2i}(\mathcal{B}, \bar{\mathbf{Q}}_l)) q^i$ where we use the standard \mathbf{W} -module structure on $H^*(\mathcal{B}, \bar{\mathbf{Q}}_l)$. For any $z \in \mathcal{W}$ let $x_z \in G$ be such that $x_z^{-1} F(x_z) = z$ and let $\mathcal{T}_z = x_z \mathcal{T} x_z^{-1}$, an F -stable maximal torus of G . For any $w \in \mathbf{W}$ we have

$$\Pi_G(w) = (-1)^{\underline{l}(w)} |G^F| q^{-\nu_G} |\mathcal{T}_w|^{-1}.$$

2.2. We assume that \mathbf{k} is as in 2.1, that G is almost simple, simply connected of exceptional type and that $K \subsetneq \Delta$. Let $\gamma \in \mathcal{S}_K$. Let $C \in \underline{\mathbf{W}}_{el}$ and let $w \in C_{min}$. We show that the condition that $\mathfrak{B}_w^\gamma \neq \emptyset$ can be tested by performing a computer calculation. We will also see that this condition depends only on K , not on γ .

We choose an \mathbf{F}_q -rational structure on G as in 2.1. We can assume that $g^{q-1} = 1$ for some/any $g \in \gamma$. Then γ is F -stable and $\gamma \cap \mathcal{T} = \gamma \cap \mathcal{T}^F$ is a single \mathcal{W} -orbit. Let $\zeta \in \gamma \cap \mathcal{T}$ be such that $Z(\zeta) = G_K$. Note that G_K is defined and split over \mathbf{F}_q ,

Now the class function $\Pi_{G_K} : \mathbf{W}_K \rightarrow \mathbf{Z}$ is well defined, see 2.1. For $z \in \mathcal{W}$ we have

$$\begin{aligned} \#(h \in G^F; h^{-1}\zeta h \in \mathcal{T}_z) &= \#(h \in G^F; x_z^{-1}h^{-1}\zeta h x_z \in \mathcal{T}) \\ &= \#(h' \in G; F(h') = h'\zeta, h'^{-1}\zeta h' \in \mathcal{T}) = \#(h' \in G; F(h') = h'\zeta, h'^{-1}\zeta h' \in \gamma \cap \mathcal{T}) \\ &= |\mathcal{W}_K|^{-1} \sum_{v \in \mathcal{W}} \#(h' \in G; F(h') = h'\zeta, h'^{-1}\zeta h' = \dot{v}\zeta\dot{v}^{-1}) \\ &= |\mathcal{W}_K|^{-1} \sum_{v \in \mathcal{W}} \#(h'' \in G; F(h'') = h''\dot{v}^{-1}\zeta F(\dot{v}), h''^{-1}\zeta h'' = \zeta) \\ &= |\mathcal{W}_K|^{-1} \sum_{v \in \mathcal{W}} \#(h'' \in G_K; F(h'') = h''\dot{v}^{-1}\zeta F(\dot{v})) \\ &= |\mathcal{W}_K|^{-1} \#(v \in \mathcal{W}; \dot{v}^{-1}\zeta F(\dot{v}) \in G_K) |G_K^F| = |\mathcal{W}_K|^{-1} \#(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K) |G_K^F|. \end{aligned}$$

(We set $h' = hx_z$; then we set $h'^{-1}\zeta h' = \dot{v}\zeta\dot{v}^{-1}$ with $v \in \mathcal{W}$; then we set $h'' = h'\dot{v}$ and we use Lang's theorem in G_K .)

As in [L5, 1.2(a)] the number of fixed points of $F : \mathfrak{B}_w^\gamma \rightarrow \mathfrak{B}_w^\gamma$, $(g, B) \mapsto (F(g), F(B))$, is given by

$$(a) \quad |(\mathfrak{B}_w^\gamma)^F| = |\mathbf{W}|^{-1} \sum_{E, E' \in \text{Irr } \mathbf{W}, z \in \mathbf{W}, g \in \gamma^F} \text{tr}(T_w, E_q)(\rho_E : R_{E'}) \text{tr}(z, E') \text{tr}(\zeta, R^1(z)).$$

(Notation of *loc.cit.*) Using [DL, 7.2] we see that

$$\begin{aligned} \text{tr}(\zeta, R^1(z)) &= \#(h \in G^F; h^{-1}\zeta h \in \mathcal{T}_z) |\mathcal{T}_z|^{-1} q^{-\nu_{G_K}} (-1)^{l(z)} \\ &= |\mathcal{W}_K|^{-1} \#(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K) |G_K^F| |\mathcal{T}_z|^{-1} q^{-\nu_{G_K}} (-1)^{l(z)} \\ &= |\mathcal{W}_K|^{-1} \#(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K) \Pi_{G_K}(v^{-1}zv). \end{aligned}$$

(We have used that the restriction to $\mathcal{W}_K = \mathbf{W}_K$ of the function $z \mapsto (-1)^{l(z)}$ on $\mathcal{W} = \mathbf{W}$ is the analogous function defined in terms of G_K .) Substituting this into (a) we obtain

$$\begin{aligned} |(\mathfrak{B}_w^\gamma)^F| &= |\gamma^F| |\mathbf{W}|^{-1} \sum_{E, E' \in \text{Irr } \mathbf{W}, z \in \mathbf{W}} \text{tr}(T_w, E_q)(\rho_E : R_{E'}) \text{tr}(z, E') \\ &\quad \times |\mathcal{W}_K|^{-1} \#(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K) \Pi_{G_K}(v^{-1}zv) \\ &= |\gamma^F| |\mathbf{W}|^{-1} \sum_{E, E' \in \text{Irr } \mathbf{W}, z \in \mathbf{W}} \text{tr}(T_w, E_q)(\rho_E : R_{E'}) \text{tr}(z, E') \text{tr}(z, \text{ind}_{\mathcal{W}_K}^{\mathcal{W}}(\Pi_{G_K})). \end{aligned}$$

Hence

$$(b) \quad |(\mathfrak{B}_w^\gamma)^F| = |G^F|/|G_K^F| \sum_{E, E' \in \text{Irr } \mathbf{W}} \text{tr}(T_w, E_q)(\rho_E : R_{E'})(E' : \Pi_{G_K})_{\mathbf{W}_K}.$$

Here $(E' : \Pi_{G_K})_{\mathbf{W}_K}$ is the inner product of Π_{G_K} (viewed as a representation of \mathbf{W}_K) with the restriction of E' to \mathbf{W}_K . We can also write (b) as follows:

$$|(\mathfrak{B}_w^\gamma)^F| = |G^F|/|G_K^F| \sum A_{E,C} \phi_{E,E'} m_{E',E''} t_{E'',K}$$

where the sum is taken over all E, E' in $\text{Irr } \mathbf{W}$, $E'' \in \text{Irr } \mathbf{W}_K$ and the notation is as follows. For $C' \in \underline{\mathbf{W}}_{el}$, $E \in \text{Irr } \mathbf{W}$ we set $A_{E,C'} = \text{tr}(T_z, E_q)$ where $z \in C'_{min}$. (Note that $A_{E,C'}$ is well defined by [GP, 8.2.6(b)].) For $E, E' \in \text{Irr } \mathbf{W}$ let $\phi_{E,E'} = (\rho_E : R_{E'})$. (Notation of [L5, 1.2].) For $E' \in \text{Irr } \mathbf{W}$, $E'' \in \text{Irr } \mathbf{W}_K$ let $m_{E',E''}$ be the multiplicity of E'' in $E'|_{\mathbf{W}_K}$. For $E'' \in \text{Irr } \mathbf{W}_K$ let $t_{E'',K}$ be the multiplicity of E'' in Π_{G_K} . Thus $|(\mathfrak{B}_w^\gamma)^F|$ is $|G^F|/|G_K^F|$ times the C -entry of the vector

$${}^t(A_{E,C'})(\phi_{E,E'})(m_{E',E''})(t_{E'',K}).$$

Here the matrix $(A_{E,C'})$ is known from the works of Geck and Geck-Michel (see [GP, 11.5.11]) and is available through the CHEVIE package [C]. The matrix $\phi_{E,E'}$ has as entries the coefficients of the "nonabelian Fourier transform" in [L1, 4.15]. The matrix $(m_{E',E''})$ ("Induction table") and the vector $(t_{E'',K})$ ("Fake degree") are available through the CHEVIE package. Thus $|(\mathfrak{B}_w^\gamma)^F|$ can be obtained by calculating the product of several explicitly known matrices. The calculation was done using the CHEVIE package. It turns out that $|(\mathfrak{B}_w^\gamma)^F|$ is a polynomial in q with integer coefficients denoted by P_C^K (it depends only on K, C not on γ, w). Note that $\mathfrak{B}_w^\gamma \neq \emptyset$ if and only if $P_C^K \neq 0$ as a polynomial in q . Thus the condition that $\mathfrak{B}_w^\gamma \neq \emptyset$ can be tested. Moreover for each K such that $P_C^K \neq 0$ and for $\gamma \in \mathcal{S}_K$, the condition that γ is C -small is equivalent to the condition that $\dim(G_K) = d_C$; in this case we have $P_C^K = m_C^K |G(\mathbf{F}_q)|$ as polynomials in q where m_C^K is an integer ≥ 1 independent of q . (For any K such that $P_C^K \neq 0$ we have $\deg(P_C^K) \geq \dim(G)$ (by [L5, 5.2]). Note that m_C^K is equal to the number of connected components of \mathfrak{B}_w^γ for $\gamma \in \mathcal{S}_K$, $w \in C_{min}$. This number can be > 1 ; in one example in type E_8 it is 10.

2.3. In this subsection we give (in the setup of 2.2) tables which describe for each exceptional type and each $C \in \underline{\mathbf{W}}_{el}$ (with one exception) some proper subsets K of Δ such that $P_C^K \neq 0$ and $\dim(G_K) = d_C$. The elements of $\Delta - \{\alpha_0\}$ are denoted by numbers $1, 2, 3, \dots$ as in [GP, p.20]. We write 0 instead of α_0 . We specify K by marking each element of $\Delta - K$ by \bullet . An element $C \in \underline{\mathbf{W}}_{el}$ is specified by indicating the characteristic polynomial of an element of C acting on $R_{\mathbf{W}}$, a product of cyclotomic polynomials Φ_d (an exception is type F_4 when there are two choices for C with characteristic polynomial $\Phi_2^2 \Phi_6$ in which case we use

the notation $(\Phi_2^2\Phi_6)'$, $(\Phi_2^2\Phi_6)''$ for what in [GP, p.407] is denoted by D_4 , $C_3 + A_1$). The notation $d; C; \chi; (K_1)_{m_1}; (K_2)_{m_2}; \dots$ means that $C \in \mathbf{W}_{el}$, $d = d_C$, $\chi = \rho_{\gamma_C}$ (γ_C as in 0.4), and K_1, K_2, \dots are proper subsets of Δ such that $P_C^{K_i} \neq 0$ and $\dim(G_{K_i}) = d_C$; we have $m_i = m_C^{K_i}$. (We omit m_i whenever $m_i = 1$.)

The notation for irreducible representations of \mathbf{W} (of type E_6, E_7, E_8) is as in [Sp]; for type F_4 it is as in [L1]; for type G_2 , 1_0 is the unit representation, 2_1 is the reflection representation and 2_2 is the other two dimensional irreducible representation of \mathbf{W} .

Type G_2 ; Δ is (012)

$$\begin{aligned} 2; \Phi_6; 1_0; & (\bullet \bullet \bullet) \\ 4; \Phi_3; 2_1; & (\bullet 1 \bullet); (\bullet \bullet 2) \\ 6; \Phi_2^2; 2_2; & (0 \bullet 2) \end{aligned}$$

Type F_4 ; Δ is (01234)

$$\begin{aligned} 4; \Phi_{12}; 1_1; & (\bullet \bullet \bullet \bullet \bullet) \\ 6; \Phi_8; 4_2; & (\bullet 1 \bullet \bullet \bullet) \\ 8; \Phi_6^2; 9_1; & (0 \bullet 2 \bullet \bullet); (\bullet 1 \bullet 3 \bullet)_2 \\ 10; (\Phi_2^2\Phi_6)'; 8_1; & (\bullet \bullet \bullet 34) \\ 10; (\Phi_2^2\Phi_6)''; 8_3; & (\bullet 12 \bullet \bullet); (0 \bullet 2 \bullet 4) \\ 12; \Phi_4^2; 12_1; & (\bullet \bullet 23 \bullet)_3; (\bullet 12 \bullet 4); (0 \bullet \bullet 34) \\ 14; \Phi_2^2\Phi_4; 16_1; & (0 \bullet 23 \bullet) \\ 16; \Phi_3^2; 6_1; & (01 \bullet 34) \\ 24; \Phi_2^4; 9_4; & (0 \bullet 234) \end{aligned}$$

$$\begin{aligned} \textit{Type } E_6; \Delta \text{ is } & \begin{pmatrix} 1 & 3 & 4 & 5 & 6 \\ & 2 & & & \\ & & 0 & & \\ & & & \bullet & \bullet & \bullet & \bullet \\ & & & & \bullet & & \\ & & & & & \bullet & \end{pmatrix} \\ 6; \Phi_3\Phi_{12}; 1_0; & \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \\ 8; \Phi_9; 6_1; & \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ & \bullet & & & \\ & & \bullet & & \\ & & & \bullet & \\ & & & & \bullet \end{pmatrix} \\ 12; \Phi_3\Phi_6^2; 30_3; & \begin{pmatrix} 1 & 3 & \bullet & \bullet & \bullet \\ & \bullet & & & \\ & & \bullet & & \\ & & & \bullet & \\ & & & & \bullet \end{pmatrix}; \begin{pmatrix} 1 & \bullet & \bullet & 5 & \bullet \\ & 2 & & & \\ & & \bullet & & \end{pmatrix} \\ 14; \Phi_2^2\Phi_3\Phi_6; 15_4; & \begin{pmatrix} 1 & \bullet & 4 & \bullet & 6 \\ & \bullet & & & \\ & & \bullet & & \\ & & & 0 & \end{pmatrix} \\ 24; \Phi_3^3; 10_9; & \begin{pmatrix} 1 & 3 & \bullet & 5 & 6 \\ & 2 & & & \\ & & 0 & & \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \textit{Type } E_7; \Delta \text{ is } & \begin{pmatrix} 0 & 1 & 3 & 4 & 5 & 6 & 7 \\ & 2 & & & & & \\ & & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & & & \\ & & & & \bullet & & \\ & & & & & \bullet & \\ & & & & & & \bullet \end{pmatrix} \\ 7; \Phi_2\Phi_{18}; 1_0; & \begin{pmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \\ 9; \Phi_2\Phi_{14}; 7_1; & \begin{pmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \\ 11; \Phi_2\Phi_6\Phi_{12}; 27_2; & \begin{pmatrix} & 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 2 & & & & \end{pmatrix} \end{aligned}$$

[illegible]

$$\begin{aligned}
& 34; \Phi_2^2 \Phi_4 \Phi_8; 4536_{13}; \quad \begin{pmatrix} 1 & 3 & 4 & \bullet & 6 & 7 & \bullet & \bullet \\ & & & 2 & & & & \end{pmatrix}; \quad \begin{pmatrix} 1 & \bullet & 4 & 5 & \bullet & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 40; \Phi_6^4; 4480_{16}; \quad \begin{pmatrix} 1 & \bullet & 4 & 5 & 6 & 7 & \bullet & \bullet \\ & & & 2 & & & & \end{pmatrix}_{10}; \quad \begin{pmatrix} 1 & 3 & 4 & 5 & \bullet & 7 & 8 & 0 \\ & & & \bullet & & & & \end{pmatrix} \\
& 42; \Phi_2^2 \Phi_6^3; 7168_{17}; \quad \begin{pmatrix} 1 & \bullet & 4 & 5 & 6 & 7 & \bullet & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 44; \Phi_2^4 \Phi_6^2; 4200_{18}; \quad \begin{pmatrix} \bullet & 3 & 4 & 5 & \bullet & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 44; \Phi_3^2 \Phi_6^2; 3150_{18}; \quad \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & \bullet & 8 & 0 \\ & & & \bullet & & & & \end{pmatrix} \\
& 46; \Phi_2^2 \Phi_3^2 \Phi_6; 2016_{19}; \quad \begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 46; \Phi_2^2 \Phi_4^2 \Phi_6; 1344_{19}; \\
& 48; \Phi_5^2; 420_{20}; \quad \begin{pmatrix} 1 & 3 & 4 & \bullet & 6 & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 60; \Phi_4^4; 840_{26}; \quad \begin{pmatrix} 1 & 3 & 4 & 5 & \bullet & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 64; \Phi_2^6 \Phi_6; 700_{28}; \quad \begin{pmatrix} \bullet & \bullet & 4 & 5 & 6 & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 66; \Phi_2^4 \Phi_4^2; 1400_{29}; \quad \begin{pmatrix} 1 & \bullet & 4 & 5 & 6 & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix} \\
& 80; \Phi_3^4; 175_{36}; \quad \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ & & & \bullet & & & & \end{pmatrix} \\
& 120; \Phi_2^8; 50_{56}; \quad \begin{pmatrix} \bullet & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ & & & 2 & & & & \end{pmatrix}
\end{aligned}$$

2.4. We prove 0.6(i) in the case where G is almost simple, simply connected of exceptional type. Let $C \in \underline{\mathbf{W}}_{el}$. Let K be a proper subset of Δ associated to C in the tables in 2.3. We can find $\gamma \in \mathcal{S}_K$ such that

(i) any element of γ has finite order.

We show that γ is C -small. By a standard argument we are reduced to the case where \mathbf{k} is as in 2.1. In this case the calculations outlined in 2.2 show that γ is C -small, as claimed.

Next we consider for $w \in C_{min}$, the map $\pi : \mathfrak{B}_w \rightarrow G$, $(g, B) \mapsto G$. Let $\mathfrak{K} = \pi_! \bar{\mathbf{Q}}_l$. Using [L4, 14.2(a)], we see that the cohomology sheaves of \mathfrak{K} behave smoothly when restricted to $\cup_{\gamma \in \mathcal{S}_K} \gamma$ (which is one of the pieces $Y_{L, \mathfrak{s}}$ in [L3, 13.11]). Since we know that when $\gamma \in \mathcal{S}_K$ satisfies (i), some cohomology sheaf of \mathfrak{K} is non-zero on γ , it follows that for any $\gamma \in \mathcal{S}_K$, some cohomology sheaf of \mathfrak{K} is non-zero on γ ; in particular, $\mathfrak{B}_w^\gamma \neq \emptyset$. It follows that any $\gamma \in \mathcal{S}_K$ is C -good. This completes the proof of Theorem 0.6(i).

3. SPRINGER REPRESENTATIONS

3.1. In the setup of 1.2 we assume that G is the identity component of $Is(V)$. If $(1 - \kappa)Q = 0$ we identify, as in [L5, 1.5], \mathbf{W} with W , the group of all permutations of $[1, \mathbf{n}]$ that commute with the involution $i \mapsto \mathbf{n} + 1 - i$. Let W_n be the group of all permutations of $[1, 2n]$ that commute with the involution $i \mapsto 2n + 1 - i$. If $\mathbf{n} = 2n$ we have $W = W_n$; if $\mathbf{n} = 2n + 1$ we identify W with W_n (hence \mathbf{W} with W_n) by $w \mapsto w'$ where $h(w'(i)) = w(h(i))$ for $i \in [1, 2n]$ and $h(i) = i$ if $i \in [1, n]$, $h(i) = i + 1$ if $i \in [n + 1, 2n]$. As in [L1, 4.5] we write the irreducible

representations of W_n in the form $[(\lambda_1 > \lambda_2 > \cdots > \lambda_{m+1}), (\mu_1 > \mu_2 > \cdots > \mu_m)]$ where $\lambda_i, \mu_i \in \mathbf{N}$, $\sum_i \lambda_i + \sum_i \mu_i = m^2 + n$ and m is sufficiently large. For example $[(0 < 1 < 2 < \cdots < n), (1 < 2 < \cdots < n)]$ is the sign representation of W_n .

If $(1 - \kappa)Q \neq 0$ we identify as in [L5, 1.5] \mathbf{W} with W'_n , the group of even permutations in W_n . As in [L1, 4.6] we write the irreducible representations of W'_n as unordered pairs $[(\lambda_1 > \lambda_2 > \cdots > \lambda_m), (\mu_1 > \mu_2 > \cdots > \mu_m)]$ where $\lambda_i, \mu_i \in \mathbf{N}$, $\sum_i \lambda_i + \sum_i \mu_i = m^2 - m + n$ and m is sufficiently large. (There are two irreducible representations corresponding to $[(\lambda_1 > \lambda_2 > \cdots > \lambda_m), (\mu_1 > \mu_2 > \cdots > \mu_m)]$ with $\lambda_i = \mu_i$ for all i .) For example $[(1 < 2 < \cdots < n), (0 < 1 < 2 < \cdots < n - 1)]$ is the sign representation of W'_n .

3.2. Let S_n be the symmetric group in n letters. Using [L6, 5.3, 4.4(a)] we see that

(a) if $n = 2c \in 2\mathbf{N}$ then $W_c \times W'_c, S_n$ are naturally reflection subgroups of W_n and we have

$$\begin{aligned} j_{W_c \times W'_c}^{W_n}(\text{sgn}) &= j_{S_n}^{W_n}(\text{sgn}) \\ &= [(0 < 2 < 3 < 4 < \cdots < c + 1), (1 < 2 < 3 < \cdots < c)]; \end{aligned}$$

(b) if $n = 2c + 1 \in 2\mathbf{N} + 1$ then $W_c \times W'_{c+1}, S_n$ are naturally reflection subgroups of W_n and we have

$$\begin{aligned} j_{W_c \times W'_{c+1}}^{W_n}(\text{sgn}) &= j_{S_n}^{W_n}(\text{sgn}) \\ &= [(1 < 2 < 3 < 4 < \cdots < c + 1), (1 < 2 < 3 < \cdots < c)]. \end{aligned}$$

Using [L6, 6.3] we see that

(c) if $n = 2c \in 2\mathbf{N}$ then $W'_c \times W'_c$ is naturally a reflection subgroup of W'_n and we have

$$j_{W'_c \times W'_c}^{W'_n}(\text{sgn}) = [(2 < 3 < 4 < \cdots < c + 1), (0 < 1 < 2 < 3 < \cdots < c - 1)].$$

Let $p_1 \geq p_2 \geq \cdots \geq p_\sigma$ be integers ≥ 1 such that $p_1 + \cdots + p_\sigma = n$. Define $\bar{p}_1 \geq \bar{p}_2 \geq \cdots \geq \bar{p}_k$ as in 1.3. Note that $\bar{p}_1 = \sigma, k = p_1$. Define $\tilde{p}_1 \geq \tilde{p}_2 \geq \cdots \geq \tilde{p}_\sigma$ by $\tilde{p}_i = p_i - 1$ if $i \in [1, \bar{p}_k]$, $\tilde{p}_i = p_i$ if $i \in [\bar{p}_k + 1, \sigma]$.

Assuming that $k > 1$ we have $\tilde{p}_\sigma = p_\sigma \geq 1$ and using [L6, 4.4(a)] we see that

$$\begin{aligned} & \text{(d) if } \sigma = 2\tau + 1 \text{ we have} \\ & [(p_\sigma < p_{\sigma-2} + 1 < \cdots < p_3 + \tau - 1 < p_1 + \tau), (p_{\sigma-1} < p_{\sigma-3} + 1 < \cdots < p_2 + \tau - 1)] \\ & = j_{S_{\bar{p}_k} \times W_{n-\bar{p}_k}}^{W_n}(\text{sgn} \boxtimes [(\tilde{p}_\sigma < \tilde{p}_{\sigma-2} + 1 < \cdots < \tilde{p}_3 + \tau - 1 < \tilde{p}_1 + \tau), \\ & (\tilde{p}_{\sigma-1} < \tilde{p}_{\sigma-3} + 1 < \cdots < \tilde{p}_2 + \tau - 1)]); \end{aligned}$$

(e) if $\sigma = 2\tau$ we have

$$\begin{aligned} & [(0 < p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \cdots < p_3 + \tau - 1 < p_1 + \tau), \\ & (p_\sigma < p_{\sigma-2} + 1 < \cdots < p_2 + \tau - 1)] \\ & = j_{S_{\bar{p}_k} \times W_{n-\bar{p}_k}}^{W_n}(\text{sgn} \boxtimes [(0 < \tilde{p}_{\sigma-1} + 1 < \tilde{p}_{\sigma-3} + 2 < \cdots < \tilde{p}_3 + \tau - 1 < \tilde{p}_1 + \tau), \\ & (\tilde{p}_\sigma < \tilde{p}_{\sigma-2} + 1 < \cdots < \tilde{p}_2 + \tau - 1)]). \end{aligned}$$

Assuming that $k > 1$, $\sigma = 2\tau$ we have $\tilde{p}_\sigma = p_\sigma \geq 1$, $n - \bar{p}_k \geq 2$ and using [L6, 6.2(a)] we see that

$$\begin{aligned}
 & [(p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \cdots < p_3 + \tau - 1 < p_1 + \tau), \\
 & (p_\sigma - 1 < p_{\sigma-2} < \cdots < p_4 + \tau - 3 < p_2 + \tau - 2)] \\
 & = j_{S_{\bar{p}_k} \times W'_{n-\bar{p}_k}}^{W'_n} (\text{sgn} \boxtimes [(\tilde{p}_{\sigma-1} + 1 < \tilde{p}_{\sigma-3} + 2 < \cdots < \tilde{p}_1 + \tau), \\
 (f) \quad & (\tilde{p}_\sigma - 1 < \tilde{p}_{\sigma-2} < \cdots < \tilde{p}_4 + \tau - 3 < \tilde{p}_2 + \tau - 2)]).
 \end{aligned}$$

We show that

$$(g) \ j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times S_{\bar{p}_1}}^{W_n} (\text{sgn}) \text{ is equal to}$$

$$[(p_\sigma < p_{\sigma-2} + 1 < \cdots < p_3 + \tau - 1 < p_1 + \tau), (p_{\sigma-1} < p_{\sigma-3} + 1 < \cdots < p_2 + \tau - 1)]$$

if $\sigma = 2\tau + 1$ and to

$$\begin{aligned}
 & [(0 < p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \cdots < p_3 + \tau - 1 < p_1 + \tau), \\
 & (p_\sigma < p_{\sigma-2} + 1 < \cdots < p_2 + \tau - 1)]
 \end{aligned}$$

if $\sigma = 2\tau$. We argue by induction on k . If $k = 1$ we have $\sigma = n$ and the result follows from (a),(b). If $k > 1$ then

$$j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times S_{\bar{p}_1}}^{W_n} (\text{sgn}) = j_{S_{\bar{p}_k} \times W_{n-\bar{p}_k}}^{W_n} (\text{sgn} \boxtimes j_{S_{\bar{p}_{k-1}} \times \cdots \times S_{\bar{p}_2} \times S_{\bar{p}_1}}^{W_{n-\bar{p}_k}} (\text{sgn}))$$

and the result follows from (d),(e) using the induction hypothesis and the transitivity of the j -induction.

We write $\bar{p}_1 = a + b$ where $b - a \in \{0, 1\}$. We show that

$$(h) \ j_{S_{\bar{p}'_k} \times \cdots \times S_{\bar{p}'_2} \times W_a \times W'_b}^{W_n} (\text{sgn}) \text{ is equal to}$$

$$[(p_\sigma < p_{\sigma-2} + 1 < \cdots < p_3 + \tau - 1 < p_1 + \tau), (p_{\sigma-1} < p_{\sigma-3} + 1 < \cdots < p_2 + \tau - 1)]$$

if $\sigma = 2\tau + 1$ and to

$$\begin{aligned}
 & [(0 < p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \cdots < p_3 + \tau - 1 < p_1 + \tau), \\
 & (p_\sigma < p_{\sigma-2} + 1 < \cdots < p_2 + \tau - 1)]
 \end{aligned}$$

if $\sigma = 2\tau$.

Using (a),(b) and the transitivity of j -induction we see that

$$\begin{aligned}
 j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times W_a \times W'_b}^{W_n} (\text{sgn}) &= j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times W_{\bar{p}_1}}^{W_n} (j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times W_a \times W'_b}^{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times W_{\bar{p}_1}} (\text{sgn})) \\
 &= j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times W_{\bar{p}_1}}^{W_n} (j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times S_{\bar{p}_1}}^{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times W_{\bar{p}_1}} (\text{sgn})) = j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times S_{\bar{p}_1}}^{W_n} (\text{sgn})
 \end{aligned}$$

and it remains to use (g).

Assuming that $\sigma = 2\tau$ we show that

$$j_{S_{\bar{p}_k} \times \dots \times S_{\bar{p}_2} \times W'_{\bar{p}_1/2} \times W'_{\bar{p}_1/2}}^{W'_n}(\text{sgn}) = [(p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \dots < p_1 + \tau),$$

(i) $(p_\sigma - 1 < p_{\sigma-2} < \dots < p_4 + \tau - 3 < p_2 + \tau - 2)]$.

We argue by induction on k . If $k = 1$ we have $\sigma = n$ and the result follows from (c). If $k > 1$ then the left hand side of (i) is equal to

$$j_{S_{\bar{p}_k} \times W'_{n-\bar{p}_k}}^{W'_n}(\text{sgn} \boxtimes j_{S_{\bar{p}_{k-1}} \times \dots \times S_{\bar{p}_2} \times W'_{\bar{p}_1/2} \times W'_{\bar{p}_1/2}}^{W'_{n-\bar{p}_k}}(\text{sgn}))$$

and the result follows from (f) using the induction hypothesis and the transitivity of the j -induction.

3.3. Assume that we are in the setup of 1.3. Let $p_* = (p_1 \geq p_2 \geq \dots \geq p_\sigma)$ be as in 1.3. We consider a unipotent class γ in G such that any $u \in \gamma$ has Jordan blocks of sizes

(i) $2p_1, 2p_2, \dots, 2p_\sigma$.

We set $\sigma = 2\tau + \kappa_\sigma$. We show:

$$\begin{aligned} \rho_\gamma &= [(0 < p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \dots < p_3 + \tau - 1 < p_1 + \tau), \\ &\quad (p_\sigma < p_{\sigma-2} + 1 < \dots < p_4 + \tau - 2 < p_2 + \tau - 1)] \text{ if } \kappa_\sigma = 0, \\ \rho_\gamma &= [(p_\sigma < p_{\sigma-2} + 1 < \dots < p_3 + \tau - 1 < p_1 + \tau), \\ \text{(a)} \quad &\quad (p_{\sigma-1} < p_{\sigma-3} + 1 < \dots < p_4 + \tau - 2 < p_2 + \tau - 1)] \text{ if } \kappa_\sigma = 1. \end{aligned}$$

To the partition (i) we will apply the procedure of [L2, 11.6]. Let $M = \sigma + \kappa_\sigma$. Let $z_M \geq \dots \geq z_2 \geq z_1$ be the sequence (i) if $\kappa_\sigma = 0$ and $2p_1, 2p_2, \dots, 2p_\sigma, 0$ if $\kappa_\sigma = 1$. The sequence $z'_M > \dots > z'_2 > z'_1$ in *loc.cit.* is

$$2p_1 + \sigma - 1, 2p_2 + \sigma - 2, \dots, 2p_\sigma \text{ (if } \kappa_\sigma = 0),$$

$$2p_1 + \sigma, 2p_2 + \sigma - 1, \dots, 2p_\sigma + 1, 0 \text{ (if } \kappa_\sigma = 1).$$

This contains $M/2$ even numbers $2y_{M/2} > \dots > 2y_2 > 2y_1$ given by

$$\{2p_t + \sigma - t; t \in [1, \sigma], \kappa_t = 0\} \text{ (if } \kappa_\sigma = 0),$$

$$\{2p_t + \sigma - t + 1; t \in [1, \sigma], \kappa_t = 0\} \sqcup \{0\} \text{ (if } \kappa_\sigma = 1)$$

and $M/2$ odd numbers $2y'_{M/2} + 1 > \dots > 2y'_2 + 1 > 2y'_1 + 1$ given by

$$\{2p_t + \sigma - t + \kappa_\sigma; t \in [1, \sigma], \kappa_t = 1\}.$$

Thus, the sets $(\{y'_{M/2} > \dots > y'_2 > y'_1\}, \{y_{M/2} > \dots > y_2 > y_1\})$ are given by

$$(\{p_t + \tau - (t+1)/2; t \in [1, \sigma], \kappa_t = 1\}, \{p_t + \tau - t/2; t \in [1, \sigma], \kappa_t = 0\}) \text{ (if } \kappa_\sigma = 0)$$

$$(\{p_t + \tau + (1-t)/2; t \in [1, \sigma], \kappa_t = 1\}, \{p_t + \tau - t/2 + 1; t \in [1, \sigma], \kappa_t = 0\} \sqcup \{0\})$$

(if $\kappa_\sigma = 1$).

If $\kappa_\sigma = 0$, the multisets

$$(\{y'_\tau - (\tau - 1) \geq \dots \geq y'_2 - 1 \geq y'_1 \geq 0\}, \{y_\tau - (\tau - 1) > \dots > y_2 - 1 > y_1\})$$

are given by

$$(\{p_1 \geq p_3 \geq \cdots \geq p_{\sigma-1} \geq 0\}, \{p_2 \geq p_4 \geq \cdots \geq p_{\sigma}\}).$$

If $\kappa_{\sigma} = 1$, the multisets

$$(\{y'_{\tau+1} - \tau \geq \cdots \geq y'_2 - 1 \geq y'_1 \geq 0\}, \{y_{\tau+1} - \tau \geq \cdots \geq y_2 - 1 \geq y_1\})$$

are given by

$$(\{p_1 \geq p_3 \geq \cdots \geq p_{\sigma} \geq 0\}, \{p_2 \geq p_4 \geq \cdots \geq p_{\sigma-1} \geq 0\}).$$

Now (a) follows from [L2, §12]. Using (a) and 3.2(g) we see that

$$(b) \quad \rho_g = j_{S_{\bar{p}_k} \times \cdots \times S_{\bar{p}_2} \times S_{\bar{p}_1}}^{W_n}(\text{sgn}).$$

3.4. Assume that we are in the setup of 1.4. Let $p_* = (p_1 \geq p_2 \geq \cdots \geq p_{\sigma})$ be as in 1.4. Define $\psi : [1, \sigma] \rightarrow \{-1, 0, 1\}$ by $\psi(t) = 1$ if t is odd and $p_{t-1} > p_t$ (the last condition is regarded as satisfied when $t = 1$); $\psi(t) = -1$ if t is even and $p_t > p_{t+1}$ (the last condition is regarded as satisfied when $t = \sigma$); $\psi(t) = 0$ for all other t . We set $\sigma = 2\tau + \kappa_{\sigma}$. If $\mathbf{n} = 2n$ we assume that $\sigma = 2\tau$. We consider a unipotent class γ in G such that any $u \in \gamma$ has Jordan blocks of sizes

- (i) $2p_1 + \psi(1), 2p_2 + \psi(2), \dots, 2p_{\sigma} + \psi(\sigma)$ if $\mathbf{n} = 2n$ (hence $\kappa_{\sigma} = 0$),
- (ii) $2p_1 + \psi(1), 2p_2 + \psi(2), \dots, 2p_{\sigma} + \psi(\sigma)$ if $\mathbf{n} = 2n + 1$ and $\kappa_{\sigma} = 1$,
- (iii) $2p_1 + \psi(1), 2p_2 + \psi(2), \dots, 2p_{\sigma} + \psi(\sigma), 1$ if $\mathbf{n} = 2n + 1$ and $\kappa_{\sigma} = 0$.

We show:

$$\begin{aligned} \rho_{\gamma} &= [(p_{\sigma} - 1 < p_{\sigma-2} < \cdots < p_4 + \tau - 3 < p_2 + \tau - 2), \\ &\quad (p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \cdots < p_3 + \tau - 1 < p_1 + \tau)], \text{ in case (i),} \\ \rho_{\gamma} &= [(p_{\sigma} < p_{\sigma-2} + 1 < \cdots < p_3 + \tau - 1 < p_1 + \tau), \\ &\quad (p_{\sigma-1} < p_{\sigma-3} + 1 < \cdots < p_4 + \tau - 2 < p_2 + \tau - 1)] \text{ in case (ii),} \\ \rho_{\gamma} &= [0 < p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \cdots < p_3 + \tau - 1 < p_1 + \tau), \\ (a) \quad &\quad (p_{\sigma} < p_{\sigma-2} + 1 < \cdots < p_4 + \tau - 2 < p_2 + \tau - 1)] \text{ in case (iii).} \end{aligned}$$

To the partition (i),(ii) or (iii) we will apply the procedure of [L2, 11.7]. Let $z_M \geq \cdots \geq z_2 \geq z_1$ be the sequence (i),(ii) or (iii) (where $M = \sigma$ in cases (i),(ii) and $M = \sigma + 1$ in case (iii)). The sequence $z'_M > \cdots > z'_2 > z'_1$ in *loc.cit.* is

$$\begin{aligned} &2p_1 + \psi(1) + \sigma - 1, 2p_2 + \psi(2) + \sigma - 2, \dots, 2p_{\sigma} + \psi(\sigma) \text{ (in cases (i),(ii)),} \\ &2p_1 + \psi(1) + \sigma, 2p_2 + \psi(2) + \sigma - 1, \dots, 2p_{\sigma} + \psi(\sigma) + 1, 1 \text{ (in case (iii)).} \end{aligned}$$

This contains $[M/2]$ even numbers $2y_{[M/2]} > \cdots > 2y_2 > 2y_1$ given by

$$\begin{aligned} &\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\} \text{ in case (i),} \\ &\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t \neq \kappa_{\psi(t)}\} \text{ in case (ii),} \\ &\{2p_t + \psi(t) + \sigma - t + 1; t \in [1, \sigma], \kappa_t \neq \kappa_{\psi(t)}\} \text{ in case (iii),} \end{aligned}$$

and $[(M+1)/2]$ odd numbers $2y'_{[(M+1)/2]} + 1 > \cdots > 2y'_2 + 1 > 2y'_1 + 1$ given by

$$\begin{aligned} &\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t \neq \kappa_{\psi(t)}\} \text{ in case (i),} \\ &\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\} \text{ in case (ii),} \\ &\{2p_t + \psi(t) + \sigma - t + 1; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\} \sqcup \{1\} \text{ in case (iii).} \end{aligned}$$

Thus, the sets $(\{y'_{[(M+1)/2]} > \cdots > y'_2 > y'_1\}, \{y_{[M/2]} > \cdots > y_2 > y_1\})$ are given

by

$$\begin{aligned}
& (\{p_t + \tau + (\psi(t) - t - 1)/2; t \in [1, \sigma], \kappa_t \neq \kappa_{\psi(t)}\}, \\
& \{p_t + \tau + (\psi(t) - t)/2; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\}) \\
& = (\{p_t + \tau + (-t - 2)/2; t \in [1, \sigma], \psi(t) = -1, \kappa_t = 0\} \\
& \sqcup \{p_t + \tau + (-t - 1)/2; t \in [1, \sigma], \psi(t) = 0, \kappa_t = 1\}, \\
& \{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 1, \kappa_t = 1\} \\
& \sqcup \{p_t + \tau + (-t)/2; t \in [1, \sigma], \psi(t) = 0, \kappa_t = 0\}) \\
& = (\{p_t + \tau + (-t - 2)/2; t \in [1, \sigma], \psi(t) = -1, \kappa_t = 0\} \\
& \sqcup \{p_{t'} + \tau + (-t' - 2)/2; t' \in [1, \sigma], \psi(t') = 0, \kappa_{t'} = 0\}, \\
& \{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 1, \kappa_t = 1\} \\
& \sqcup \{p_{t'} + \tau + (1 - t')/2; t' \in [1, \sigma], \psi(t') = 0, \kappa_{t'} = 1\}) \\
& = (\{p_t + \tau + (-t - 2)/2; t \in [1, \sigma], \kappa_t = 0\} \\
& \sqcup \{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \kappa_t = 1\})
\end{aligned}$$

in case (i),

$$\begin{aligned}
& (\{p_t + \tau + (\psi(t) - t)/2; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\}, \\
& \{p_t + \tau + (\psi(t) + 1 - t)/2; t \in [1, \sigma], \kappa_t \neq \kappa_{\psi(t)}\}) \\
& = (\{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 1, \kappa_t = 1\} \\
& \sqcup \{p_t + \tau - t/2; t \in [1, \sigma], \psi(t) = 0, \kappa_t = 0\}, \\
& \{p_t + \tau - t/2; t \in [1, \sigma], \psi(t) = -1, \kappa_t = 0\} \\
& \sqcup \{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 0, \kappa_t = 1\}) \\
& = (\{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 1, \kappa_t = 1\} \\
& \sqcup \{p_{t'} + \tau + (1 - t')/2; t' \in [1, \sigma], \psi(t') = 0, \kappa_{t'} = 1\}, \\
& \{p_t + \tau - t/2; t \in [1, \sigma], \psi(t) = -1, \kappa_t = 0\} \\
& \sqcup \{p_{t'} + \tau - t'/2; t' \in [1, \sigma], \psi(t') = 0, \kappa_{t'} = 0\}) \\
& = (\{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \kappa_t = 1\}, \\
& \{p_t + \tau - t/2; t \in [1, \sigma], \kappa_t = 0\})
\end{aligned}$$

in case (ii),

$$\begin{aligned}
& (\{p_t + \tau + (\psi(t) - t)/2; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\} \sqcup \{0\}, \\
& \{p_t + \tau + (\psi(t) + 1 - t)/2; t \in [1, \sigma], \kappa_t \neq \kappa_{\psi(t)}\}) = \\
& (\{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 1, \kappa_t = 1\} \sqcup \{0\} \\
& \sqcup \{p_t + \tau - t/2; t \in [1, \sigma], \psi(t) = 0, \kappa_t = 0\} \sqcup \{0\}, \\
& \{p_t + \tau - t/2; t \in [1, \sigma], \psi(t) = -1, \kappa_t = 0\} \\
& \sqcup \{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 0, \kappa_t = 1\}) \\
& = (\{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \psi(t) = 1, \kappa_t = 1\} \sqcup \{0\} \\
& \sqcup \{p_{t'} + \tau + (1 - t')/2; t' \in [1, \sigma], \psi(t') = 0, \kappa_{t'} = 1\} \sqcup \{0\}, \\
& \{p_t + \tau - t/2; t \in [1, \sigma], \psi(t) = -1, \kappa_t = 0\} \\
& \sqcup \{p_{t'} + \tau - t'/2; t' \in [1, \sigma], \psi(t') = 0, \kappa_{t'} = 0\}) \\
& = (\{p_t + \tau + (1 - t)/2; t \in [1, \sigma], \kappa_t = 1\} \sqcup \{0\}, \\
& \{p_t + \tau - t/2; t \in [1, \sigma], \kappa_t = 0\})
\end{aligned}$$

in case (iii). (We have used that, if $\kappa_t = 0$, $t < \sigma$ and $\psi(t) = 0$, then $ps(t+1) = \psi(t)$, $p_{t+1} = p_t$; if $\kappa_t = 1$ and $\psi(t) = 0$ then $\psi(t-1) = \psi(t)$, $p_{t-1} = p_t$.)

In case (i) the multisets

$$(\{y'_\tau - \tau + 1 \geq \cdots \geq y'_2 - 1 \geq y'_1\}, \{y_\tau - \tau + 1 \geq \cdots > y_2 - 1 \geq y_1\})$$

are given by

$$(\{p_2 - 1 \geq p_4 - 1 \geq \cdots \geq p_\sigma - 1\}, \{p_1 + 1 \geq p_3 + 1 \geq \cdots \geq p_{\sigma-1} + 1\}).$$

In case (ii) the multisets

$$(\{y'_{\tau+1} - \tau \geq \cdots \geq y'_2 - 1 \geq y'_1\}, \{y_\tau - \tau + 1 \geq \cdots \geq y_2 - 1 \geq y_1\})$$

are given by

$$(\{p_1 \geq p_3 \geq \cdots \geq p_\sigma\}, \{p_2 \geq p_4 \geq \cdots \geq p_{\sigma-1}\}).$$

In case (iii) the multisets

$$(\{y'_{\tau+1} - \tau \geq \cdots \geq y'_2 - 1 \geq y'_1\}, \{y_\tau - \tau + 1 \geq \cdots \geq y_2 - 1 \geq y_1\})$$

are given by

$$(\{p_1 \geq p_3 \geq \cdots \geq p_{\sigma-1} \geq 0\}, \{p_2 \geq p_4 \geq \cdots \geq p_\sigma\}).$$

Now (a) follows from [L2, §13]. Using (a) and 3.2(i),(h), we see that

$$\rho_\gamma = j_{S_{\bar{p}_k}^n \times \cdots \times S_{\bar{p}_2} \times W'_{\bar{p}_1/2} \times W'_{\bar{p}_1/2}}^{W'_n}(\text{sgn}) \text{ in case (i);}$$

$$\rho_\gamma = j_{S_{\bar{p}_k}^n \times \cdots \times S_{\bar{p}_2} \times W_{(\bar{p}_1 - \kappa_\sigma)/2} \times W'_{(\bar{p}_1 + \kappa_\sigma)/2}}^{W_n}(\text{sgn})$$

(b) in case (ii),(iii).

3.5. We prove Theorem 0.6(ii). If G is of type A the result is immediate. If G is of classical type other than A , the result follows from 3.3(b) and 3.4(b). If G is of exceptional type, the result follows from the tables in 2.3 using the data on j -induction available in the CHEVIE package [C]. This completes the proof of Theorem 0.6.

3.6. Let w_{p_*} be the conjugacy class in W_n (as in 3.1) associated to $p_* = (p_1 \geq p_2 \geq \cdots \geq p_\sigma)$ (where $p_1 + p_2 + \cdots + p_\sigma = n$) as in [L5, 1,6]. We can view w_{p_*} as an element of \mathbf{W} in the cases where G is as in 1.2 with either $Q = 0, \kappa = 0$ or with $Q \neq 0, \kappa = 1$. In both cases w_{p_*} has minimal length in its conjugacy class C which is in $\underline{\mathbf{W}}_{el}$. Let γ_C, γ'_C be the corresponding C -small unipotent classes (one is in a symplectic group, one is in an odd orthogonal group). From 3.3(b), 3.4(b), we see, using 3.2(g),(h), that

(a) *the Springer representations $\rho_{\gamma_C}, \rho_{\gamma'_C}$ are the same.*

We see that the map $C \mapsto \rho_{\gamma_C}$ from $\underline{\mathbf{W}}_{el}$ to $\text{Irr}(\mathbf{W})$ depends only on the Weyl group \mathbf{W} , not on the underlying root system.

REFERENCES

- [DL] P.Deligne and G.Lusztig, *Representations of reductive groups over finite fields*, Ann.Math. **103** (1976), 103-161.
- [C] M.Geck, G.Hiss, F.Lübeck, G.Malle and G.Pfeiffer, *A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras*, Appl. Algebra Engrg. Comm. Comput. **7** (1996), 1175-210.
- [GP] M.Geck and G.Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, Clarendon Press Oxford, 2000.
- [L1] G.Lusztig, *Characters of reductive groups over a finite field*, Ann.Math. Studies 107, Princeton U.Press, 1984.
- [L2] G.Lusztig, *Intersection cohomology complexes on a reductive group*, Inv.Math. **75** (1984), 205-272.
- [L3] G.Lusztig, *Character sheaves, I*, Adv.Math. **56** (1985), 193-237.
- [L4] G.Lusztig, *Character sheaves, III*, Adv.Math. **57** (1985), 266-315.
- [L6] G.Lusztig, *Unipotent classes and special Weyl group representations*, J.Algebra **321** (2009), 3418-3449.
- [L5] G.Lusztig, *From conjugacy classes in the Weyl group to unipotent classes*, arXiv:1003.0412.
- [Sp] N.Spaltenstein, *On the generalized Springer correspondence for exceptional groups*, Algebraic groups and related topics, Adv.Stud.Pure Math.6, North Holland and Kinokuniya, 1985, pp. 317-338.

DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE, MA 02139